

Electromagnetic scattering from large aspect ratio lossy dielectric solids in a conducting medium

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Abstract

The spheroidal T-matrix formalism developed by Hackman¹⁻³ and Sammelmann⁴⁻⁶ for acoustic scattering is extended to electromagnetic scattering from lossy dielectric solids in a conducting medium. The spheroidal T-matrix formalism exhibits superior performance with respect to the spherical T-matrix formalism for objects that deviate appreciably from a spherical shape. Both acoustic (elastic) and electromagnetic scattering are solutions of the vector Helmholtz equation. In the case of elastic wave scattering, the displacement field has 3 degrees of freedom corresponding to the 2 polarization states of the shear wave and the longitudinal mode. In the case of electromagnetic scattering, the electric (magnetic) field has 2 polarization states corresponding to left and right-handed photons, but lacks a longitudinal mode. The T-matrix description of electromagnetism mimics the T-matrix description of elastic wave scattering in the absence of a longitudinal mode. Indeed, the stress tensor of the displacement is replaced by the exterior derivative of the electric field in Betti's identity in the derivation of the T-matrix formalism of scattering from a lossy dielectric solid. Continuity of the displacement and surface traction is replaced by continuity of the tangential components of the electric and magnetic fields in the boundary conditions. In the case of a time harmonic field, the presence of a finite conductivity in the medium is represented by the insertion of an imaginary component of the wavenumber that is proportional to the conductivity in the medium. In the case of complex wavenumber, the Helmholtz equation is no longer a self-adjoint operator, and the S-matrix is no longer unitary. This article describes some of the features unique to scattering in seawater due to the large conductivity of the medium.

1.0 INTRODUCTION

This article describes the application of the spheroidal T-matrix formalism developed by Hackman¹⁻³ and Sammelmann⁴⁻⁶ to electromagnetic scattering from isotropic lossy dielectric solids in a conducting medium.

Section 2.0 begins with an introduction to the scalar and vector spheroidal wavefunctions, followed by a derivation of the generalization of Betti's third identity

for electromagnetic scattering, and concluding with a derivation of the T-matrix for a large aspect ratio dielectric solid and a perfectly conducting solid based upon the generalization of Betti's identity. Section 3 concludes.

2.0 SPHEROIDAL T-MATRIX FORMALISM

The spheroidal T-matrix description of electromagnetic scattering from a conducting lossy dielectric solid in a conducting medium is derived in this section. The symbols ε_+ , μ_+ , σ_+ , and ε_- , μ_- , σ_- are used to denote the electric susceptibility, magnetic permeability, and conductivity of the media exterior and interior to the scatterer, where the symbols

$$\begin{aligned} k_+ &= \sqrt{\omega^2 \varepsilon_+ \mu_+ + i\omega \mu_+ \sigma_+} \\ k_- &= \sqrt{\omega^2 \varepsilon_- \mu_- + i\omega \mu_- \sigma_-} \end{aligned} \quad (2.1)$$

are the wavenumber in the exterior and interior regions, respectively.

The notation of Morse and Feshbach⁷ is used to denote the regular and irregular spheroidal radial functions of the third kind by the symbols $je_{ml}(c|\xi)$ and $he_{ml}(c|\xi)$, respectively. These functions correspond to the radial functions of the first and third kinds denoted by $R_{ml}^{(1)}(c|\xi)$ and $R_{ml}^{(3)}(c|\xi)$ by Flammer⁸ and Abramowitz and Stegun⁹. The symbols $S_{ml}(c|\eta)$ and $S_{ml}^{(2)}(c|\eta)$ are used to denote the angular function of the first and second kind. Here, $c = kf$ is the dimensionless wave number, where f is the semi-focal distance of the spheroidal coordinate system. The normalization convention of Flammer⁸ for the expansion coefficients $d_k^{ml}(c)$ of the angular functions of the first kind is adopted.

The regular and irregular basis functions of the scalar Helmholtz equation in the exterior and interior of the scatterer are defined as follows.

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$$\begin{aligned} \text{Re} \psi_{\sigma ml}^{(\pm)} &= j e_{ml}(c_{\pm} | \xi) S_{\sigma ml}^{(\pm)}(c_{\pm} | \eta, \varphi) \\ \psi_{\sigma ml}^{(\pm)} &= h e_{ml}(c_{\pm} | \xi) S_{\sigma ml}^{(\pm)}(c_{\pm} | \eta, \varphi) \end{aligned} \quad (2.2)$$

The angular functions

$$\begin{aligned} S_{\sigma ml}^{(\pm)}(c_{\pm} | \eta, \varphi) &= \\ \sqrt{\frac{\varepsilon(m)}{2\pi\Lambda_{ml}(c_{\pm})}} S_{ml}(c_{\pm} | \eta) &\begin{cases} \cos(m\varphi), \sigma = e \\ \sin(m\varphi), \sigma = o \end{cases} \end{aligned} \quad (2.3)$$

$$\begin{aligned} &\int_{-1}^{+1} d\eta S_{ml}^{*(\pm)}(c_{\pm} | \eta) S_{ml}^{(\pm)}(c_{\pm} | \eta) \\ &= \delta_{l,l'}^l \Lambda_{ml}(c_{\pm}) \end{aligned}$$

are the spheroidal equivalent of the spherical harmonics. Note that in the above definition of the norm of the angular functions the integral of the angular function and its complex conjugate is computed. For complex wavenumbers, the above integral without the complex conjugate is equal to a unitary matrix, since the angular wave equation is non-self-adjoint in the case of complex wavenumbers, and the corresponding eigenvalues are in general complex. This is to be contrasted with the case of the spherical wave functions where the angular wave equation is independent of frequency and hence self-adjoint even for complex wavenumbers. The corresponding radial wave equation is superficially non-self-adjoint, but it is conformally equivalent to a wave equation that is independent of frequency and self-adjoint by a simple rescaling of the radial coordinate. Hence the eigenvalues for the spherical radial wave equation are also real. This difference between the behavior of the spheroidal and spherical wave functions for complex frequencies has resulted in a number of errors in the literature, and subtle differences in various identities and expansions for real and complex frequencies in the spheroidal case. For example, in the case of complex wavenumber, the scalar Green's function of the Helmholtz equations is no longer given by the simple bilinear summation in terms of the regular and irregular spheroidal wavefunctions.

$$\begin{aligned} G(\vec{r}, \vec{r}') &= \exp[+ik | \vec{r} - \vec{r}' |] / 4\pi | \vec{r} - \vec{r}' | \\ &\neq ik \sum_{\sigma ml} \psi_{\sigma ml} \text{Re} \psi_{\sigma ml} \end{aligned} \quad (2.4)$$

Rather, it is given by a bilinear expansion of the following form.

$$\begin{aligned} G(\vec{r}, \vec{r}') &= \exp[+ik | \vec{r} - \vec{r}' |] / 4\pi | \vec{r} - \vec{r}' | \\ &= ik \sum_{\sigma ml, l'} \psi_{\sigma ml} U_{l,l'}^m \text{Re} \psi_{\sigma ml'} \end{aligned} \quad (2.5)$$

The matrix $U_{l,l'}^m$ is a symmetric, unitary matrix

$((U^{-1})_{l,l'}^m = (U^*)_{l',l}^m)$, which reduces to the identity matrix in the limit the imaginary component of the wavenumber, vanishes, that is, $\text{Im}(c) \rightarrow 0$.

Define the following basis $\{V_{\tau, \sigma ml}^{(\pm)}\}$ for the vector Helmholtz equation in spheroidal coordinates in the exterior and interior regions of the scatterer.

$$\begin{aligned} V_{1, \sigma ml}^{(\pm)} &= \vec{\nabla} \times (\vec{a} \psi_{\sigma ml}^{(\pm)}) \\ V_{2, \sigma ml}^{(\pm)} &= \frac{1}{k_{\pm}} \vec{\nabla} \times (V_{1, \sigma ml}^{(\pm)}) \\ V_{3, \sigma ml}^{(\pm)} &= \frac{1}{k_{\pm}} \vec{\nabla} \psi_{\sigma ml}^{(\pm)} \end{aligned}$$

(2.6)

$$\vec{a} = \vec{r}$$

$$(a_{\xi}, a_{\varphi}, a_{\eta}) = \begin{cases} (f^2 \xi, 0, +f^2 \eta), & \text{prolate} \\ (f^2 \xi, 0, -f^2 \eta), & \text{oblate} \end{cases}$$

The first pair of vector basis functions is a pair of solutions of the transverse vector Helmholtz equation, and the third basis function is a solution of the longitudinal vector Helmholtz equation. Since there is no longitudinal component of the electric and magnetic field, the vector index τ is restricted to the values 1 and 2 corresponding to the 2 transverse modes.

Define the following angular functions, as the generalization of the vector spherical harmonics.

$$\begin{aligned} A_{1, \sigma ml}^{(\pm)} &= (\hat{\phi} \sqrt{1-\eta^2} \partial_{\eta} - \hat{\eta} \frac{1}{\sqrt{1-\eta^2}} \partial_{\varphi}) S_{\sigma ml}^{(\pm)} \\ A_{2, \sigma ml}^{(\pm)} &= (\hat{\eta} \sqrt{1-\eta^2} \partial_{\eta} + \hat{\phi} \frac{1}{\sqrt{1-\eta^2}} \partial_{\varphi}) S_{\sigma ml}^{(\pm)} \\ A_{3, \sigma ml}^{(\pm)} &= \hat{\xi} S_{\sigma ml}^{(\pm)} \end{aligned} \quad (2.7)$$

The above functions are not solutions of the vector Helmholtz equation. They satisfy the following orthogonality condition, where the upper sign is used for prolate coordinates and the lower sign for oblate coordinates.

$$\begin{aligned}
\oint d\varphi d\eta A_{1,\sigma ml} \bullet A_{1,\sigma' m' l'} &= \delta_{\sigma}^{\sigma'} \delta_m^{m'} \Omega_{l,l'}^{1,m} \\
\oint d\varphi d\eta A_{2,\sigma ml} \bullet A_{2,\sigma' m' l'} &= \delta_{\sigma}^{\sigma'} \delta_m^{m'} \Omega_{l,l'}^{2,m} \\
\oint d\varphi d\eta A_{3,\sigma ml} \bullet A_{3,\sigma' m' l'} &= \delta_{\sigma}^{\sigma'} \delta_m^{m'} \Omega_{l,l'}^{3,m} \\
\oint d\varphi d\eta A_{1,\sigma ml} \bullet A_{3,\sigma' m' l'} &= 0 \\
\oint d\varphi d\eta A_{2,\sigma ml} \bullet A_{3,\sigma' m' l'} &= 0 \\
\oint d\varphi d\eta A_{1,\sigma ml} \bullet A_{2,\sigma' m' l'} &= 0 \\
\Omega_{l,l'}^{1,m} &= \Omega_{l,l'}^{2,m} = \\
\frac{1}{\sqrt{\Lambda_{ml} \Lambda_{ml'}}} \int_{-1}^{+1} d\eta (1-\eta^2) \partial_{\eta} S_{ml}(c, \eta) \partial_{\eta} S_{ml'}(c, \eta) \\
+ \frac{1}{\sqrt{\Lambda_{ml} \Lambda_{ml'}}} \int_{-1}^{+1} d\eta \frac{m^2}{(1-\eta^2)} S_{ml}(c, \eta) S_{ml'}(c, \eta) \\
\Omega_{l,l'}^{3,m} &= \frac{1}{\sqrt{\Lambda_{ml} \Lambda_{ml'}}} \int_{-1}^{+1} d\eta S_{ml}(c, \eta) S_{ml'}(c, \eta)
\end{aligned} \tag{2.8}$$

The matrix $\Omega_{l,l'}^{\tau,m}$ is a symmetric unitary matrix in the case of complex wavenumber. In the case of real wavenumber they reduce to the following matrices.

$$\begin{aligned}
\Omega_{l,l'}^{1,m} &= \Omega_{l,l'}^{2,m} = \lambda_{ml} \delta_l^{l'} \\
\mp \frac{c^2}{\sqrt{\Lambda_{ml} \Lambda_{ml'}}} \int_{-1}^{+1} d\eta \eta^2 S_{ml}(c, \eta) S_{ml'}(c, \eta)
\end{aligned} \tag{2.9}$$

$$\Omega_{l,l'}^{3,m} = \delta_l^{l'}$$

Here, λ_{ml} are the eigenvalues of the spheroidal angular (radial) wave equation.

At this point it is worth noting that the vector basis functions $\{V_{\tau,\sigma ml}^{(\pm)}\}$ have the following asymptotic behavior for large radial coordinate $\xi \rightarrow \infty$.

$$\begin{aligned}
V_{1,\sigma ml} &\rightarrow he_{ml}(c, \xi) A_{1,\sigma ml} \\
V_{2,\sigma ml} &\rightarrow \frac{1}{c} \partial_{\xi} he_{ml}(c, \xi) A_{2,\sigma ml} \\
V_{3,\sigma ml} &\rightarrow \frac{1}{c} \partial_{\xi} he_{ml}(c, \xi) A_{3,\sigma ml}
\end{aligned} \tag{2.10}$$

This limit coupled with Betti's identity is useful in proving various orthogonality relations between the vector basis functions.

The generalization of Betti's identity for acoustics scattering to electromagnetic scattering follows.

Theorem: Let u and v be a pair of arbitrary solutions of the source-free, transverse, vector Helmholtz equation in the region V .

$$-\vec{\nabla} \times \vec{\nabla} \times \vec{u} + k^2 \vec{u} = 0 \tag{2.11}$$

Define the exterior derivative $T(u)_{\alpha\beta}$ and its normal component $\vec{t}(u)$ as follows.

$$\begin{aligned}
T(u)_{\alpha\beta} &= \nabla_{\alpha} u_{\beta} - \nabla_{\beta} u_{\alpha} \\
\vec{t}(u)_{\alpha} &= n^{\sigma} T(u)_{\sigma\alpha}
\end{aligned} \tag{2.12}$$

Here \hat{n} is the outward going normal to the boundary of V . Then the following surface integral over the boundary of V vanishes.

$$\oint dA \{ \vec{u} \bullet \vec{t}(v) - \vec{t}(u) \bullet \vec{v} \} \tag{2.13}$$

The above theorem follows from Green's theorem and the fact that the transverse wave equation can be recast in the following form in terms of the exterior derivative.

$$(-\vec{\nabla} \times \vec{\nabla} \times \vec{u} + k^2 \vec{u})_{\alpha} = \nabla^{\sigma} T(u)_{\sigma\alpha} + k^2 u_{\alpha} = 0 \tag{2.14}$$

Here, the exterior derivative $T(u)_{\alpha\beta}$ plays the role of the stress tensor in elastic wave scattering, and the interior derivative $\vec{t}(u)$ is analogous to the surface traction of the stress tensor.

The following is a useful corollary of the above theorem.

Corollary: Let S_0 be an arbitrary closed, compact surface about the origin, S_{∞} the sphere at infinity, and V the volume whose boundary consists of the union of these two surfaces. Then given an arbitrary pair of solutions of the source-free, transverse, vector Helmholtz equation in the region V , the following integrals over the surfaces S_0 and S_{∞} are equal.

$$\int_{S_0} dA \{ \vec{u} \bullet \vec{t}(v) - \vec{t}(u) \bullet \vec{v} \} = \int_{S_{\infty}} dA \{ \vec{u} \bullet \vec{t}(v) - \vec{t}(u) \bullet \vec{v} \} \tag{2.15}$$

The previous theorem and its corollary allows one to write the following orthogonality relations for the

vector basis function for an arbitrary compact, closed surface.

$$\iint_S dA \{ t(\text{Re } V_{\tau, \sigma ml}) \bullet \text{Re } V_{\tau', \sigma' m' l'} - \text{Re } V_{\tau, \sigma ml} \bullet t(\text{Re } V_{\tau', \sigma' m' l'}) \} = 0$$

$$\iint_S dA \{ t(V_{\tau, \sigma ml}) \bullet V_{\tau', \sigma' m' l'} - V_{\tau, \sigma ml} \bullet t(V_{\tau', \sigma' m' l'}) \} = 0$$

$$\begin{aligned} & \iint_S dA \{ t(\text{Re } V_{\tau, \sigma ml}) \bullet V_{\tau', \sigma' m' l'} - \text{Re } V_{\tau, \sigma ml} \bullet t(V_{\tau', \sigma' m' l'}) \} \\ &= \begin{cases} -i/k(+i)^{l-l'} \delta_{\sigma}^{\sigma'} \delta_m^{m'} \delta_{\tau}^{\tau'} \Omega_{l, l'}^{\tau, m}, \tau \neq 3 \\ 0, \tau = 3 \text{ or } \tau' = 3 \end{cases} \end{aligned} \quad (2.16)$$

The above orthogonality relationships follow from the previous corollary, where the asymptotic limit of the vector basis functions $\{V_{\tau, \sigma ml}\}$ in the limit $\xi \rightarrow \infty$, and the orthogonality conditions of the vector spheroidal harmonics $\{A_{\tau, \sigma ml}\}$ are used to evaluate the surface integral over the sphere at infinity.

At this point the machinery necessary to derive the spheroidal T-matrix description of electromagnetic scattering from a lossy dielectric solid has been introduced. In the following, the indices n and n' are used to collectively designate the indices $\tau, \sigma ml$ and $\tau', \sigma' m' l'$, respectively. In general, the index τ is restricted to the values of 1 or 2, since the exterior derivative of the third vector basis function vanishes identically.

The incident and scattered electric fields have the following expansions in terms of the regular and irregular vector basis functions in the region exterior to the scatterer.

$$\begin{aligned} \vec{E}_{inc} &= \sum_n a_n \text{Re } \psi_n^{(+)} \\ \vec{E}_{sc} &= \sum_n f_n \psi_n^{(+)} \end{aligned} \quad (2.17)$$

Using the above orthogonality conditions for the vector basis functions and the above expansions for the incident and scattered fields, one arrives at the following linear equations for the incident and scattered fields.

$$\begin{aligned} (Nf)_n &= \sum_{n'} N_{n, n'} f_{n'} \\ &= + \int_S \{ \text{Re } V_n^{(+)} \bullet t(E_+) - t(\text{Re } V_n^{(+)}) \bullet E_+ \} \end{aligned} \quad (2.18a)$$

$$\begin{aligned} (N^T a)_n &= \sum_{n'} N_{n', n} a_{n'} \\ &= - \int_S \{ V_n^{(+)} \bullet t(E_+) - t(V_n^{(+)}) \bullet E_+ \} \\ N_{\tau \sigma ml, \tau' \sigma' m' l'} &= \frac{i}{k_+} \delta_{\sigma}^{\sigma'} \delta_m^{m'} \delta_{\tau}^{\tau'} (+i)^{l-l'} \Omega_{l, l'}^{\tau, m} \end{aligned} \quad (2.18b)$$

$$N^T = +N$$

Here, E_+ denotes the limit as the exterior electric field approaches the surface, and E_- denotes the limit as the interior electric field approaches the surface.

Next, one applies Betti's identity to the interior surface fields. Here the assumption that the interior is source free is made, and the interior electric field can be expanded in terms of the regular basis functions. In this case the following surface integral vanishes, since Betti's identity applied to a pair of regular vector basis functions vanishes.

$$\int_S \{ \text{Re } \psi_n^{(-)} \bullet t(E_-) - t(\text{Re } \psi_n^{(-)}) \bullet E_- \} = 0 \quad (2.19)$$

There are 3 equations for the interior and exterior surface fields. The following boundary conditions are utilized to rewrite these 3 equations in a more tractable form.

$$\begin{aligned} \hat{n} \times \vec{E}_+ &= \hat{n} \times \vec{E}_- \\ \hat{n} \times \vec{H}_+ &= \hat{n} \times \vec{H}_- \end{aligned} \quad (2.20)$$

Here, the source-free Maxwell's equations are utilized to relate the magnetic field to the curl of the electric field.

$$\vec{H} = \frac{1}{i\omega\mu} \vec{\nabla} \times \vec{E} \quad (2.21)$$

Using the above relationship between the electric and magnetic field, the original boundary conditions can be rewritten in the following form.

$$\begin{aligned}\hat{n} \times \vec{E}_+ &= \hat{n} \times \vec{E}_- \\ \frac{1}{\mu_+} \hat{n} \times (\vec{\nabla} \times \vec{E}_+) &= \frac{1}{\mu_-} \hat{n} \times (\vec{\nabla} \times \vec{E}_-)\end{aligned}\quad (2.22)$$

Using the above boundary conditions, coupled with the following vector identities,

$$\begin{aligned}\vec{t}(E) &= -\hat{n} \times (\vec{\nabla} \times \vec{E}) \\ \vec{E} \bullet t(u) &= -\vec{E} \bullet (\hat{n} \times (\vec{\nabla} \times \vec{u})) = -(\vec{E} \times \hat{n}) \bullet (\vec{\nabla} \times \vec{u})\end{aligned}\quad (2.23)$$

one can re-express equations 2.18 and 2.19 in terms of the surface fields E_- , and $t(E_+)$.

$$\begin{aligned}(Nf)_n &= \sum_{n'} N_{n,n'} f_{n'} \\ &= + \int_S \{ \text{Re } V_n^{(+)} \bullet t(E_+) - t(\text{Re } V_n^{(+)} \bullet E_- \} \\ (N^T a)_n &= \sum_{n'} N_{n',n} a_{n'} \\ &= - \int_S \{ V_n^{(+)} \bullet t(E_+) - t(V_n^{(+)} \bullet E_- \}\end{aligned}\quad (2.24)$$

$$\int_S \{ \text{Re } \psi_n^{(-)} \bullet t(E_+) \frac{\mu_-}{\mu_+} - t(\text{Re } \psi_n^{(-)} \bullet E_- \} = 0$$

The surface fields E_- , and $t(E_+)$ can be expressed by the following expansion in terms of the vector spheroidal harmonics.

$$\begin{aligned}E_- &= \sum_n \alpha_n A_n^{(-)} \\ t(E_+) &= \sum_n \beta_n A_n^{(+)}\end{aligned}\quad (2.25)$$

Substituting these expansions into equations 2.24, we arrive at the following 3 linear algebraic equations.

$$\begin{aligned}Nf &= -\text{Re } Q\alpha + \text{Re } M\beta \\ Na &= +Q\alpha - M\beta \\ 0 &= R\alpha - P\beta\end{aligned}\quad (2.26)$$

Here the matrices are defined as the following surface integrals, where there are no longer any unknowns appearing in the integrals.

$$N_{\tau\sigma ml, \tau'\sigma'm'l'} = \frac{i}{k_+} \delta_{\sigma}^{\sigma'} \delta_m^{m'} \delta_{\tau}^{\tau'} (+i)^{l-l'} \Omega_{l,l'}^{\tau,m}(c_+)$$

$$\begin{aligned}\text{Re } Q_{n,n'} &= \int_S t(\text{Re } V_n^{(+)} \bullet A_{n'}^{(-)}) \\ Q_{n,n'} &= \int_S t(V_n^{(+)} \bullet A_{n'}^{(-)})\end{aligned}$$

$$\begin{aligned}\text{Re } M_{n,n'} &= \int_S \text{Re } V_n^{(+)} \bullet A_{n'}^{(+)} \\ M_{n,n'} &= \int_S V_n^{(+)} \bullet A_{n'}^{(+)}\end{aligned}$$

$$R_{n,n'} = \int_S t(\text{Re } V_n^{(-)} \bullet A_{n'}^{(-)})$$

$$P_{n,n'} = \frac{\mu_-}{\mu_+} \int_S \text{Re } V_n^{(-)} \bullet A_{n'}^{(+)} \quad (2.27)$$

The solution of equations 2.26 for the scattered field is of the following form.

$$f = Ta \quad (2.28)$$

T is the spheroidal T-matrix given below.

$$T = -N^{-1}(\text{Re } QR^{-1}P - \text{Re } M)(QR^{-1}P - M)^{-1}N \quad (2.29)$$

This form is analogous to the form developed by Hackman³ to describe the acoustic scattering from an elastic solid.

In order to obtain an expression for the interior field in terms of the incident field, we make the following expansion of the interior field in terms of the regular basis functions in the interior of the target.

$$E_{\text{int}} = \sum_n b_n \text{Re } V_n^{(-)} \quad (2.30)$$

Next, Betti's identity is used to obtain the following relationship between this expansion and the surface fields.

$$\sum_{n'} N_{n',n}^{(-)} b_{n'} = \int_S \{ t(V_n^{(-)} \bullet E_- - V_n^{(-)} \bullet t(E_-) \} \quad (2.31)$$

$$N_{\tau\sigma ml, \tau'\sigma'm'l'}^{(-)} = \frac{i}{k_-} \delta_{\sigma}^{\sigma'} \delta_m^{m'} \delta_{\tau}^{\tau'} (+i)^{l-l'} \Omega_{l,l'}^{\tau,m}(c_-)$$

Apply the boundary condition,

$$t(E_-) = \frac{\mu_-}{\mu_+} t(E_+) \quad (2.32)$$

to equation 2.31 to obtain the following expression for this linear equation in terms of the surface fields,

E_- and $t(E_+)$.

$$\sum_{n'} N_{n',n}^{(-)} b_{n'} = \int_S \{ t(V_n^{(-)}) \cdot E_- - \frac{\mu_-}{\mu_+} V_n^{(-)} \cdot t(E_+) \} \quad (2.33)$$

Insert the expansions given in equations 2.25 for the surface fields into equation 2.33 to obtain the following linear equation for the interior field.

$$N^{(-)} b = B \alpha - C \beta$$

$$B_{n,n'} = \int_S t(V_n^{(-)}) \cdot A_{n'}^{(+)} \quad (2.34)$$

$$C_{n,n'} = \frac{\mu_-}{\mu_+} \int_S V_n^{(-)} \cdot A_{n'}^{(+)}$$

Solving equations 2.26 and 2.34, we obtain the following expression for the interior field in terms of the incident field.

$$b = T^{(-)} a \quad (2.35)$$

$$T^{(-)} = (N^{(-)})^{-1} (B R^{-1} P - C) (Q R^{-1} P - M)^{-1} N$$

In the case of a perfectly conducting solid, the boundary condition is that the tangential components of the electric field vanish at the surface.

$$\hat{n} \times \vec{E}_+ = \hat{n} \times \vec{E}_- = 0 \quad (2.36)$$

The discontinuity in the tangential component of the magnetic field is proportional to the induced surface current. In this case the T-matrix for a perfectly conducting solid is given by the following expression.

$$T = -N^{-1} \text{Re } M (M)^{-1} N \quad (2.37)$$

3.0 CONCLUSIONS

The spheroidal T-matrix for a lossy dielectric solid in a conducting medium was derived from a generalization of Betti's identity to electromagnetic scattering. The resulting T-matrix is formally identical to

the T-matrix for elastic scattering from an elastic solid, where the stress tensor is replaced by the exterior derivative of the electric field.

Future work will focus on the effects of boundaries on the scattering from proud, buried, and partially buried targets.

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